

# A NOTE OF THE CONVERSE OF SCHUR'S THEOREM

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**ABSTRACT.** Let  $G$  be an arbitrary group such that  $G/Z(G)$  is finite, where  $Z(G)$  denotes the center of the group  $G$ . Then  $\gamma_2(G)$ , the commutator subgroup of  $G$ , is finite. This result is known as Schur's theorem. The motive of this short note is to provide a quick survey on the converse of Schur's theorem and to give some further remarks. Let  $Z_2(G)$  denote the second center of a group  $G$ . Then we point out that a converse of Schur's theorem can be formulated as follows: If  $\gamma_2(G)$  is finite and  $Z_2(G)/Z(G)$  is finitely generated, then  $G/Z(G)$  is finite. Moreover,  $G$  is isoclinic (in the sense of P. Hall) to a finite group.

## 1. INTRODUCTION AND SOME HISTORY

Let  $G$  be an arbitrary group. Let  $Z(G)$ ,  $Z_2(G)$ ,  $\gamma_2(G)$  denote the center, the second center and the commutator subgroup of  $G$ . Let  $K(G)$  denote the set of all commutators of  $G$  and for  $x \in G$ ,  $[x, G]$  denote the set  $\{[x, g] \mid g \in G\}$ . Notice that  $|[x, G]| = |x^G|$ , where  $x^G$  denote the conjugacy class of  $x$  in  $G$ . If  $[x, G] \subseteq Z(G)$ , then  $[x, G]$  becomes a subgroup of  $G$ . Exponent of a subgroup  $H$  of  $G$  is denoted by  $\exp(H)$ . For a subgroup  $H$  of  $G$ ,  $C_G(H)$  denotes the centralizer of  $H$  in  $G$  and for an element  $x \in G$ ,  $C_G(x)$  denotes the centralizer of  $x$  in  $G$ .

Understanding the relationship between  $\gamma_2(G)$  and  $G/Z(G)$  goes back at least to 1904 when I. Schur [8] proved that the finiteness of  $G/Z(G)$  implies the finiteness of  $\gamma_2(G)$ . A natural question which arises here is about the converse of Schur's theorem, i.e., whether the finiteness of  $\gamma_2(G)$  implies the finiteness of  $G/Z(G)$ . Unfortunately the answer is negative as can be seen for infinite extraspecial  $p$ -group for an odd prime  $p$ . But there has been attempts to modify the statement and get conclusions. On one hand people studied the situation by putting some extra conditions on the group. For example, B. H. Neumann [4, Corollary 5.41] proved that  $G/Z(G)$  is finite if  $\gamma_2(G)$  is finite and  $G$  is finitely generated. Moreover, he proved that if  $G$  is generated by  $k$  elements, then  $|G/Z(G)| \leq |\gamma_2(G)|^k$ . This result is recently generalized by P. Niroomand [6] by proving that  $G/Z(G)$  is finite if  $\gamma_2(G)$  is finite and  $G/Z(G)$  is finitely generated. Niroomand's result is further generalized by B. Sury [9] by getting the same conclusion on the assumption that  $K(G)$  is finite and  $G/Z(G)$  is finitely generated. B. H. Neumann [5, Theorem 3.1] proved the following result: Let  $G$  be an arbitrary group. Then the lengths of the conjugacy classes of elements of  $G$  are bounded above by a finite natural number if and only if  $\gamma_2(G)$  is finite. A special case of this result was re-proved by Sury on the way to generalizing Niroomand's result.

On the other hand, somewhat weaker conclusion were obtained by assuming the finiteness of the commutator subgroup. For example, it follows from a result of

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P. Hall [2] that  $G/Z_2(G)$  is finite if  $\gamma_2(G)$  is finite. Explicit bounds on the order of  $G/Z_2(G)$  were first given by I. D. Macdonald [3, Theorem 6.2] and later on improved by S. Podoski and B. Szegedy [7] by showing that if  $|\gamma_2(G)/(\gamma_2(G) \cap Z(G))| = n$ , then  $|G/Z_2(G)| \leq n^{c \log_2 n}$  with  $c = 2$ . These are really very striking results which suggest to look for the obstruction in the direction of the converse of Schur's theorem. And it is very surprizing (at least for the people who didn't know P. Hall's result) to observe that all these obstructions, which stop  $G/Z(G)$  to be finite, lie between  $Z(G)$  and  $Z_2(G)$ . The following theorem is a modification (perhaps the finest one) of the converse of Schur's theorem with a minimal condition on  $G$  along with the finiteness of  $\gamma_2(G)$ .

**Theorem A.** *Let  $G$  be an arbitrary group such that  $Z_2(G)/Z(G)$  is finitely generated and  $\gamma_2(G)$  is finite. Then  $G/Z(G)$  is finite.*

Let  $G$  be a group as in Theorem A. Since  $\gamma_2(G)$  is finite, it follows from a result of P. Hall [2] (as mentioned above) that  $G/Z_2(G)$  is finite. Now the supposition that  $Z_2(G)/Z(G)$  is finitely generated, shows that  $G/Z(G)$  is finitely generated. So the proof of Theorem A now follows from the main theorem of P. Niroomand [6].

## 2. SOME FURTHER REMARKS

We first remark that Theorem A also follows from the following theorem which provides an upper bound on the size of  $G/Z(G)$  in terms of  $|\gamma_2(G)Z(G)/Z(G)|$ , the rank of  $Z_2(G)/Z(G)$  and exponents of certain sets of commutators (here these sets are really subgroups of  $G$ ) of representatives of generators of  $Z_2(G)/Z(G)$  with the elements of  $G$ .

**Theorem B.** *Let  $G$  be an arbitrary group. Let  $|\gamma_2(G)Z(G)/Z(G)| = n$  is finite and  $Z_2(G)/Z(G)$  is finitely generated by  $x_1Z(G), x_2Z(G), \dots, x_tZ(G)$  such that  $\exp([x_i, G])$  is finite for  $1 \leq i \leq t$ . Then*

$$|G/Z(G)| \leq n^{2 \log_2 n} \prod_{i=1}^t \exp([x_i, G]).$$

*Proof.* Let  $G$  be a group as in Theorem B. Then it follows from Theorem 1 of [7] that  $|G/Z_2(G)| \leq n^{2 \log_2 n}$ . Now by the given hypothesis  $\exp([x_i, G])$  is finite for all  $i$  such that  $1 \leq i \leq t$ . Suppose that  $\exp([x_i, G]) = n_i$ . Since  $[x_i, G] \subseteq Z(G)$ , it follows that  $[x_i^{n_i}, G] = [x_i, G]^{n_i} = 1$ . Thus  $x_i^{n_i} \in Z(G)$  and no lesser power of  $x_i$  than  $n_i$  can lie in  $Z(G)$ . Since  $Z_2(G)/Z(G)$  is abelian, we have  $Z_2(G)/Z(G) \leq \prod_{i=1}^t \exp([x_i, G])$ . Hence

$$|G/Z(G)| = |G/Z_2(G)| |Z_2(G)/Z(G)| \leq n^{2 \log_2 n} \prod_{i=1}^t \exp([x_i, G]).$$

Proof of the theorem is now complete.  $\square$

The following concept is due to P. Hall [1]. Two groups  $K$  and  $H$  are said to be *isoclinic* if there exists an isomorphism  $\phi$  of the factor group  $\bar{K} = K/Z(K)$  onto  $\bar{H} = H/Z(H)$ , and an isomorphism  $\theta$  of the subgroup  $\gamma_2(K)$  onto  $\gamma_2(H)$  such that

the following diagram is commutative

$$\begin{array}{ccc} \bar{K} \times \bar{K} & \xrightarrow{a_G} & \gamma_2(K) \\ \phi \times \phi \downarrow & & \downarrow \theta \\ \bar{H} \times \bar{H} & \xrightarrow{a_H} & \gamma_2(H). \end{array}$$

The resulting pair  $(\phi, \theta)$  is called an *isoclinism* of  $K$  onto  $H$ . Notice that isoclinism is an equivalence relation among groups.

Our second remark is the following proposition which can be derived from Macdonald's result [3, Lemma 2.1]. But we here sketch a proof for completeness.

**Proposition 2.1.** *Let  $G$  be the group as in Theorem B. Then  $G$  is isoclinic to some finite group.*

*Proof.* Let  $G$  be the given group. Then it follows from Theorem B that  $G/Z(G)$  is finite. Hence by Schur's theorem  $\gamma_2(G)$  is finite. Now it follows from a result of P. Hall [1] that there exists a group  $H$  which is isoclinic to  $G$  and  $Z(H) \leq \gamma_2(H)$ . Since  $|\gamma_2(G)| = |\gamma_2(H)|$  is finite,  $Z(H)$  is finite. Hence  $H$  is finite.  $\square$

Now we re-state Lemma 9 of [7] with little more ingredients and mention a proof for the sake of completeness.

**Lemma 2.2.** *Let  $G$  be a group and  $H$  be a subgroup of  $G$  generated by  $h_1, h_2, \dots, h_t$  and  $Z(G)$  such that  $[h_i, G]$  is finite for  $1 \leq i \leq t$ . Then  $|G/C_G(H)| \leq \prod_{i=1}^t |[h_i, G]|$ .*

*Proof.* The following is immediate from Poincare's theorem.

$$|G : C_G(H)| \leq \prod_{i=1}^t |G : C_G(h_i)| = \prod_{i=1}^t |h_i^G|,$$

since  $|G : C_G(z)| = 1$  for all  $z \in Z(G)$ . Since  $|[h_i, G]| = |h_i^G|$ , the proof of the lemma is complete.  $\square$

Our final remark is that one can use Lemma 2.2 to improve the known bounds on the size of  $G/Z(G)$  in terms of conjugacy class lengths of non-central generators of  $G$  as follows.

**Proposition 2.3.** *Let  $G$  be an arbitrary group such that  $G/Z(G)$  is finitely generated by  $x_1 Z(G), x_2 Z(G), \dots, x_t Z(G)$  such that  $[x_i, G]$  is finite for  $1 \leq i \leq t$ . Then  $|G/Z(G)| \leq \prod_{i=1}^t |[x_i, G]|$ .*

*Proof.* Put  $H = G$  in Lemma 2.2.  $\square$

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